A PREVIEW OF ELECTRON EMISSION PHENOMENA

1. Thermal emission: over barrier with energy supplied by heating
2. Field emission: through barrier by tunneling enabled by intense electric fields
3. Photoemission: over barrier with energy supplied by $\hbar \omega$ absorption
4. Secondary emission: over barrier with energy supplied by primary $e^-$
5. Space Charge: Limitation of current crossing anode-cathode (AK) gap $D$

Electron Emission Phenomena

All are current density $J$ relations (field, thermal, space charge) or yield ratios $QE$, $\delta$ involving current density (photo, secondary)
Sections Outline

1. The Canonical Equations
   - Historical Motivation
   - Conventions
   - Basic Current Density

2. Thermal-Field Emission
   - Original GTF
   - Shape Factor
   - Improvements

3. Usage in Simulation
   - Compound Emitter
   - Hot Fibers and Nanowires
   - Rough Surface
**History: TF Emission from a Tungsten Tip**


- **Dyke et al.:** as $F = q|E|$ decreased, average $J(F, T)$ over surface (left) compared to $J(F, 300K)$, showed strong T-dependence
- **FE sources** can run **hot** at high $J$; **TE sources** can have protrusions. Breakdown sites are a combination of both.
- $10^{1.2} = 16\times$ is a substantial increase in current!
- **Question:** rapid way to predict $J(F, T)$ from a complex geometry?
Current Vs. Voltage $I(V)$:

- RLD-Like:
  
  $$R \equiv \ln\left[\frac{I(V)}{I(V_o)}\right]$$
  
  $$\sim B \sqrt{V} - B \sqrt{V_o}$$

- FN-Like ($p = 2$)
  
  $$R' \equiv \ln\left(\frac{I(V)}{V^p} \cdot \frac{V_o^p}{I(V_o)}\right)$$
  
  $$\sim \frac{B'}{V_o} - \frac{B'}{V}$$

For a sharp W tip:

$$p \rightarrow 2 - \nu \text{ with } \nu \sim 0.773$$

**Geittner, Gärtner, and Raasch.**

*Low temperature properties of Ba-dispenser cathodes.”*


“...The curves impressively demonstrate the superior emission properties of the “Sc”/Re-I cathodes (lower $e \cdot \Phi$). Compared to a pure W-I cathode, an emission gain of nearly four orders of magnitude is obtained. Moreover, superimposed to the thermionic emission both cathodes clearly show remarkable contributions by field emission..."
Image Charge Barrier: Defines $F$ and $Q$

$$U(x) = \mu + \Phi - q|\mathcal{E}|x - \frac{q^2}{16\pi\varepsilon_0 x}$$

$$\equiv \mu + \Phi - Fx - \frac{Q}{x}$$

Max of $U(x)$ defines $\phi$

$$\partial_x U(x_o) = 0 \leftrightarrow x_o = \sqrt{Q/F}$$

$$\phi \equiv \Phi - \sqrt{4QF}$$

- $F$ is a force; $U$ is an energy.
- Schottky barrier lowering factor:
  $$y(\mu) = \sqrt{4QF}/\Phi$$
- Phrasing: ‘Field” and “field-like” used for quantities dependent upon $F$.

$\alpha = \frac{q^2}{4\pi\varepsilon_0 \hbar c}$ and $[\text{nefq}] = \text{nm}, \text{eV}, \text{fs}, q = 1$

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<th>Definition</th>
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<td>Field</td>
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<td>Force $F = q\mathcal{E}$</td>
<td>q GV/m</td>
<td>1 eV/nm</td>
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<tr>
<td>Pot. Energy $U = q\phi$</td>
<td>q Volt</td>
<td>1 eV</td>
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The Canonical Equations

Thermal: Richardson-Laue-Dushman
C. Herring, M. Nichols, Rev. Mod. Phys. 21, 185 (1949).

\[ J_{RLD}(T) = A_{RLD}T^2 \exp\left(-\frac{\phi}{k_BT}\right) \] (4)

Field: Fowler Nordheim
E.L. Murphy, R.H. Good, Phys Rev 102, 1464 (1956).

\[ J_{FN}(F) = \frac{A_{FN}}{t(y)^2} F^2 \exp\left(-\nu(y) \frac{B_{FN}\Phi^{3/2}}{F}\right) \] (5)

Photo: Fowler-DuBridge

\[ QE \equiv \frac{\hbar \omega}{q} \left(\frac{J}{I_{\omega}}\right) \propto (\hbar \omega - \phi)^2 \] (6)

Secondary: Baroody

\[ \delta(E_o) = BE_o e^{-\lambda} \int_0^1 \exp(\lambda s^2) ds \] (7)

Space Charge: Child-Langmuir

\[ J_{CL}(\varphi_a) = \frac{4\varepsilon_0}{9D^2} \left(\frac{2q}{m}\right)^{1/2} \varphi_a^{3/2} \] (8)

Φ = Work Function; T = Temperature; F = qE; 
\( \hbar \omega \) = Photon energy; \( I_{\omega} \) = laser intensity;
\( E_o \) = Primary electron beam energy;
\( \lambda \) = energy loss per unit length
\( D \) = anode-cathode gap; \( \varphi_a \) = anode potential

Equations follow from \( J \) evaluation

† Part II: The Canonical Equations [https://doi.org/10.1002/9781119051794]
**General Current Density I**

- For eigenstates $\psi_k(x)$:
  \[ j_k(x, t) = \frac{\hbar}{2mi} \left\{ \psi_k^\dagger \partial_x \psi_k - \psi_k \partial_x \psi_k^\dagger \right\} \]

- Transmission probability: ratio of $j_k$
  \[ D(k) = \frac{j_{\text{trans}}(k)}{j_{\text{incident}}(k)} \quad (9) \]

- Rectangular barrier: $D(k) = |t(k)|^2$

**Supply function from FD Distribution**

\[
\beta_T = \frac{1}{k_B T} \\
E = \frac{\hbar^2}{2m} (k_x^2 + k_{\perp}^2) = E_x + E_{\perp} \\
f(k_x) = \frac{2}{2\pi^2} \int_0^\infty \frac{2\pi k_{\perp} dk_{\perp}}{1 + e^{\beta_T(E-\mu)}} = \frac{m}{\pi \beta_T \hbar^2} \ln \left[ 1 + e^{\beta_T(\mu-E_x)} \right] \quad (10)
\]

**Tsu-Esaki-like (1D) Current Density Relation with $E(k) = \hbar^2 k^2 / 2m$**

\[
J(F, T) = q\rho \frac{\hbar \langle k \rangle}{m} = \frac{q}{2\pi} \int_0^\infty \frac{\hbar k}{m} D(k) f(k) dk = \frac{q}{2\pi \hbar} \int_0^\infty D(E) f(E) dE \quad (11)
\]
Barrier Height $V_0 = \hbar^2 k_o^2 / 2m$, width $L$

\[
\frac{1}{D_{rec}(k)} \equiv 1 + \frac{k_o^4 \sinh^2 \left( L \sqrt{k_o^2 - k^2} \right)}{4k^2(k_o^2 - k^2)} \quad (12)
\]

- **Gamow Factor (aka “WKB”)**

\[
\theta(k) = 2L \sqrt{k_o^2 - k^2}
\]

\[
\rightarrow 2 \int_{x_-}^{x_+} k(x) \, dx
\quad (13)
\]

- **Kemble form of $D(k)$:**

\[
D(k) \approx \frac{1}{\{1 + \exp[\theta(k)]\}} \quad (14)
\]

- □ is hard for Kemble
  $D(k_o) \neq 1/2$; $D(k > k_o)$ oscillates
  Kemble works better for $\Delta, \cap$
Measured Total E Distributions: Voltage increased from 500 V (1.1 GV/m) to 1600 V (3.5 GV/m) $E_m$ moves from $\mu + \phi$ to $\mu$.


Exact $\theta(E_x)$ compared to linear approximations using $\beta_F(\mu)$ and $\beta_F(\mu + \phi)$ used in oGTF equation.

- $T = 850$ K, $F = 1$ eV/nm
- $\mu = 7$ eV, $\Phi = 2$ eV
- Gray $\theta = 0$ (horizontal); $E = \mu, \mu + \phi$ (vertical)
- $\theta(E)$ is well approximated by a cubic polynomial
**General Current Density IV**

Insert $f(E) +$ Kemble $D(E)$ into Tsu-Esaki

$$J(F, T) = \frac{qm}{2\pi^2\beta T^2\hbar^2} \int_0^\infty \frac{\ln \left[ 1 + \exp[\beta T(\mu - E)] \right]}{1 + \exp[\theta(E)]} dE \approx A_R L D T^2 N \left[ \frac{\beta T}{\beta_F(E_m)}, \theta(E_m) \right]$$  \hspace{1cm} (15)

$$\beta_T = 1/k_B T \quad \& \quad \beta_F = -\partial_E \theta \iff \text{Energy slope factors}$$

Factors: $\mu = 7 \text{ eV}, \Phi = 4.5 \text{ eV}, F = 2.7 \text{ eV/nm}$. (left) $T = 700 \text{ K}$; (right) $T = 1570 \text{ K}$
The Canonical Equations
Thermal-Field Emission
Usage in Simulation

Historical Motivation
Conventions
Basic Current Density

Photoemission from Semiconductors

\[ QE = (1 - R) \frac{\int_{E_a}^{\hbar \omega - E_g} EdE \int_{x_m}^{1} xdx D_\Delta \left[ E x^2 \right] f_\lambda (x, E)}{2 \int_{0}^{\hbar \omega - E_g} E \left[ \int_{0}^{1} dx \right] dE} \]  

where \( x_m = \sqrt{E_a/E} \). Tall triangular barrier: Use \( D_\Delta(E) \):

\[ D_\Delta(E) \approx \frac{4[E(E + U_o)]^{1/2}}{(E^{1/2} + (E + U_o)^{1/2})^2} \rightarrow QE \approx \frac{2Cs^5}{(1 + s^2)(1 + \sqrt{1 + s^2})(s + \sqrt{1 + s^2})^2} \]  

where \( s^2 = (\hbar \omega - E_g - E_a)/E_a \) and \( C = [(1 - R)/(1 + p)] \times \) a factor of order unity

\[ \log_{10}(QE) \]

- \( \hbar \omega \) (eV)

\[ \log_{10}(QE) \]

- \( \hbar \omega \) (eV)
Secondary Yield

Primary electrons shed their energy \( (E_o) \) as they plow through bulk material \( (dE/dx) \), losing about \( \Delta E \approx 2.5E_g \) per generation of each secondary.

Bethe Eq. \( Z = 6 \) and \( N = 0.29241 \) mole/cm\(^3\) for carbon

\[
\frac{dE}{dx} = -\frac{192\pi NQ^2}{E} \ln \left( 1 + \frac{2^{4/3} E_0}{QZ^2} \right)
\]  
(18)

Range

\[
R(E_o) = \int_{E_o}^{0} \left( \frac{dE}{dx} \right)^{-1} dE
\]  
(19)

Yield \( B \sim 1/\Delta E, \gamma = \text{exp. penetration inv. length} \)

\[
\delta(E_o) = -B \int_{0}^{R(E)} \left( \frac{dE}{dx} \right) e^{-\gamma x} dx \\
\approx BE_o e^{-\gamma R(E_o)} \frac{F_n[\gamma R(E_o)]}{[\gamma R(E_o)]^{1/n}}
\]  
(20)

\[
F_n(y) \equiv \int_{0}^{y^{1/n}} e^{sn} ds
\]  
(21)

High Secondary Yield of Diamond \(^a\)

\( \delta_m = \text{maximum yield} \)

Diamond: \( E_m = 8.275 \) eV, \( \delta_m = 183.53 \)

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The Canonical Equations
Thermal-Field Emission
Usage in Simulation

Original GTF
Shape Factor
Improvements

Original General Thermal-Field-Photoemission (oGTFP) I

Gamow Factor $\theta(E)$ linearized:

$$\theta(E) \approx \theta(E_m) + \beta_F(E_m)(E_m - E)$$  \hspace{1cm} (22)

Special Cases for image charge $U(x)$ using Elliptical Integral Functions $v(y)$ and $t(y)$

$$\theta(\mu) = \frac{4}{3hF} \sqrt{2m\Phi^3} v[y(\mu)]$$
$$\theta(\mu + \phi) = 0$$
$$\beta_F(\mu) = \frac{2}{hF} \sqrt{2m\Phi} t[y(\mu)] \equiv \frac{1}{k_B T_{\text{min}}}$$
$$\beta_F(\mu + \phi) = \frac{\pi}{hF} \sqrt{m\Phi} \sqrt{y(\mu)} \equiv \frac{1}{k_B T_{\text{max}}}$$

Extension to Photoemission:

$$\theta(E_m) \rightarrow \theta(E_m + \hbar \omega)$$
$$N(n, s) \rightarrow N(n, -s)$$  \hspace{1cm} (24)


N(n,s) Approximation

$$N(n, s) \approx e^{-s} n^2 \sum \left( \frac{1}{n} \right) + e^{-ns} \Sigma(n)$$

$$n \rightarrow \frac{\beta_T}{\beta_F(E_m)}; \quad s \rightarrow \theta(\mu)$$

$$N(1, s) = (s + 1)e^{-s}$$
$$\Sigma(x) \approx \frac{1 + x^2}{1 - x^2} - 0.36 x^2 - 0.106 x^4$$

Define $n(F, T)$ and $s(F, T)$ by

- F-like $n > 1$: $T < T_{\text{min}}$: $E_m \rightarrow \mu$
- T-like $n < 1$: $T > T_{\text{max}}$: $E_m \rightarrow \mu + \phi$
- TF-like $n = 1$: Otherwise: $\beta_T = \beta_F(E_m)$
- $s = \theta(E_m) + \beta_F(E_m)(E_m - \mu)$
General Thermal-Field current density $J_{GTF}(F, T)$

$$J_{GTF}(F, T) = \begin{cases} n^{-2}J_F + J_T & (n < 1) \\
J_F + n^2J_T & (n > 1) \end{cases}$$ (26)

The two current densities $J_F$ and $J_T$ are defined by

$$J_F = A_{RLD}(k_B\beta_F)^{-2}\Sigma\left(\frac{\beta_F}{\beta_T}\right)\exp\left[-\beta_F (E_o - \mu)\right] \quad (n \to \infty, J_F \to J_{FN})$$ (27)

$$J_T = A_{RLD}(k_B\beta_T)^{-2}\Sigma\left(\frac{\beta_T}{\beta_F}\right)\exp\left[-\beta_T (E_o - \mu)\right] \quad (n \to 0, J_T \to J_{RLD})$$ (28)

$E_o(E_m)$ is defined by

$$E_o(E_m) \equiv E_m + \frac{\theta(E_m)}{\beta_F(E_m)} \to \mu + \Phi \times \begin{cases} 2\nu(y)/3t(y) & (n \gg 1) \\
1 & (n \ll 1) \end{cases}$$ (29)

Generalized Photoemission current density $J_P(F, T) \propto QE I_\omega$

$$J_P \propto (\hbar\omega - \phi)^2 + \frac{\pi^2}{3} \left(\beta_T^{-2} + \beta_F^{-2}\right) \quad (n^2 \ll 1, J_P \to J_{FD})$$ (30)
**Conventional:**

\( J_{GTF}(F, T) \) [A/cm\(^2\)] for \( T = 1173 \) K and \( \Phi = 4.5 \) eV.

Also: RLD = Richardson, FN = Fowler-Nordheim, and gray lines corresponding to \( F(T_{\text{min}}) = 1.361 \) eV/nm and \( F(T_{\text{max}}) = 2.273 \) eV/nm as per Eq. (23)

Yellow dot = 376× Gray dot

**Low Work Function:**

\( J_{GTF}(F, T) \) [A/cm\(^2\)] for \( T = 1173 \) K and \( \Phi = 2.1 \) eV.

Also: RLD = Richardson, FN = Fowler-Nordheim, and gray lines corresponding to \( F(T_{\text{min}}) = 1.36 \) eV/nm and \( F(T_{\text{max}}) = 1.617 \) eV/nm as per Eq. (23)

Yellow dot = 7.1× Gray dot

Notice small “kinks” at \( F(T_{\text{max/min}}) \)
Reformulated General Thermal-Field (rGTF)

\( \circ \) GTF: \(^{a}\)

- Three regimes separated by \( T_{\text{max}}, T_{\text{min}} \) in Eq. (23); \( \beta_{F}(E) \) evaluated only at \( \beta_{F}(\mu) \) and \( \beta_{F}(\mu + \phi) \); \( v(y) \) and \( t(y) \) used;

- \( T > T_{\text{max}}, n \leq 1 \leftrightarrow \text{T-regime}; \)
  \( T < T_{\text{min}}, n \geq 1 \leftrightarrow \text{F-regime} \)

- \( T_{\text{min}} \leq T \leq T_{\text{max}}, n = 1 \leftrightarrow \text{TF} \)

\( \circ \) GTF: \(^{b},^{c}\)

- Schottky \( v(y), t(y) \) no longer used:
  Shape Factors \( \sigma(E), u(E) \) are always used to find \( \theta(E), \beta_{F}(E) \)

- \( E_{m} \) is always found exactly
  \( \theta(E) \) always linearized about \( E_{m} \)

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\(^{b}\) K.L. Jensen; A reformulated general thermal-field emission equation J. Appl. Phys. 126, 065302 (2019).


Redefined Regimes

- \( n < 0.95 \leftrightarrow \text{T-regime} \)
- \( n > 1.05 \leftrightarrow \text{F-regime} \)
- \( |n - 1| \leq 0.05 \leftrightarrow \text{TF-regime} \)
σ(E) and u(E) (related to \( \partial_E \theta \)) defined by

\[
\sigma(E) = \int_{x_-}^{x_+} \left\{ \frac{U(x) - E}{U_o - E} \right\}^{1/2} \frac{dx}{L} \quad (31)
\]

\[
u(E) = \int_{x_-}^{x_+} \left\{ \frac{U_o - E}{U(x) - E} \right\}^{1/2} \frac{dx}{L} \quad (32)
\]

Length/Height scales: \( (\phi = \Phi - \sqrt{4QF}) \)

\[
FL(E) = \sqrt{(\mu + \Phi - E)^2 - 4QF} \quad (33)
\]

\[
h\kappa(E) \equiv \sqrt{2m(\mu + \phi - E)}
\]

Gamow factor using Shape Function

\[
\theta(E) = 2 \sigma(E) \kappa(E)L(E) \quad (34)
\]

σ(E) is factor accounting for shape

- rectangular: \( \sigma_\square = 1 \)
- triangular: \( \sigma_\triangle = 2/3 = 0.6667 \)
- parabolic: \( \sigma_\cap = \pi/4 = 0.7854 \)

Relation to Fowler-Nordheim Equation

\[
J_{FN}(F, \Phi) = \frac{qm}{2\pi^2\hbar^3} \frac{e^{-2\sigma(y(\mu))}\kappa(\mu)L(\mu)}{[2\nu(\mu)\kappa(\mu)L(\mu)]^2}
\]

Relation to SN Functions, \( y = \sqrt{4QF}/\Phi \)

\[
\sigma(\mu) = \frac{2\nu(y)}{3(1 - y)\sqrt{1 + y}}; \quad \nu(\mu) = \frac{2\nu(y)}{\sqrt{1 + y}}
\]
\section*{Shape Factor Method II}

\textit{o}GTF relies on Schottky $v(y)$ and $t(y)$, therefore only for image charge barriers. This is a \textbf{limitation} both theoretically and computationally. \textbf{Remove them.}

\textbf{Introduce:}

\begin{itemize}
  \item $y(E) = \frac{\sqrt{4FQ}}{\mu + \Phi - E}$
  \item $\sigma(E) \rightarrow \sigma[y(E)]$ depends only on $y$
\end{itemize}

\textit{r}GTF formalism\footnote{Jensen et al., J. Appl. Phys. 126, 245301 (2019)}

\begin{itemize}
  \item $\theta(E) = 2\sigma(y)\kappa(E)L(E)$; $\beta_F(E) \equiv -\partial_E \theta(E)$
  \item $\Delta < U(x) < \cap: 2/3 < \sigma(y) < \pi/4$
  \item \textbf{Image:} $U(x) - E = F(x - a)(b - x)/x$
    Cubic $U(x) - E = \gamma(x - a)(b - x)(c - x)/x$
\end{itemize}

\textbf{Finding new terms is easy}

\begin{itemize}
  \item $L(E) = b - a \Rightarrow$ sol’ns of quadratic eq.
  \item $U(x_0)$: $x_0$ is root of $\partial_x U(x) = 0$
  \item $\kappa(E) = \sqrt{2m(U(x_0) - E)}/\hbar$
  \item $\sigma(y)$ well approximated by polynomial
\end{itemize}

\textbf{Recover FN}

Limit $y \rightarrow 0$ ($Q \rightarrow 0$) no image:

$$\lim_{Q \rightarrow 0} \beta_F(E) = \frac{2\sqrt{2m}}{\hbar F} (\mu + \Phi - E)^{1/2}$$

“F-like” because reproduces triangular Fowler Nordheim result

\textbf{Recover RLD}

Limit $y \rightarrow 1$ ($E \rightarrow \mu + \phi$):

$$\lim_{y \rightarrow 1} \beta_F(E) = \frac{\pi(2m)^{1/2}Q^{1/4}}{\hbar F^{3/4}}$$

reproduces $\cap$ barrier

“T-like” because $E_m \approx \mu + \phi$ when $T$ high and $F \rightarrow 0$. 

\textbf{a}
Shape Factor Method III

Define:

\[ V(a) = V(b) = E, \partial_x V(x_o) = 0 \]
\[ y(E) = \sqrt{4QF}/(\mu + \Phi - E) \]
\[ x_o^2 = \frac{Q}{F}; \quad L = \frac{2x_o}{y} \sqrt{1-y^2}; \quad a = \frac{x_o}{y} - \frac{L}{2} \]  

(35)

Exact Way:

\[ \sigma(y) = \sqrt{\frac{1+y}{1-y}} \mathbb{P}\left(\frac{a}{L}\right) \]  

(36)

\[ \mathbb{P}(p) \equiv \frac{1}{2} \int_0^{\pi/2} \frac{[\sin(2x)]^2}{\sqrt{(\sin x)^2 + p}} \, dx \]  

(37)

Easy Way: \( \mathbb{P}(p) \to \mathbb{P}_n(p), \ n < 24 \)

\[ \mathbb{P}_n(p) \equiv \frac{\pi}{4n} \sum_{j=1}^{n-1} \frac{[\sin(\pi j/n)]^2}{\sqrt{[\sin(\pi j/2n)]^2 + p}} \]  

(38)

End points are exact:

\( \sigma(0) = 2/3, \sigma(1) = \pi/4 \)

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<th>( x_j )</th>
<th>( y_j )</th>
<th>( \sigma(y_j) )</th>
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<td>1</td>
<td>( \pi/4 )</td>
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<td>4</td>
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<td>0</td>
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3-5 exact values of \( \sigma(y) \) can create highly accurate quadratic (good) or quartic (great) \( \sigma[y(x)] \) fits.
Maximum (Em) Evaluation

\[ E_m \] for various \( T \) with \( \mu = 7 \text{ eV}, \Phi = 4.5 \text{ eV} \).

TF-regime is where \( E_m \) departs \( \mu + \phi \) line (thermal-like) and converges with \( \mu \) line (field-like), where oGTF overestimates \( dJ(E) \) extent (left).

Observe low \( T \) (FN emission) is also overestimated because \( \beta_F(E_m) > \beta_F(\mu) \).

Near \( n \to 1 \), using \( E_m \approx \mu \) (FN) or \( E_m \approx \mu + \phi \) (para) overestimates \( dJ(E) \), causes “kinks” in \( J(F, T) \). Reformulated Approach revises procedure at \( n = 1 \).

\[ T = 850 \text{ K}, F = 1 \text{ eV/nm}, \mu = 7 \text{ eV}, \Phi = 2 \text{ eV} \]
MAXIMUM \((E_m)\) EVALUATION II

\(N(n, s)\) is approximation to “exact” \(N[\theta(E)]\):

\[
N[\theta(E)] = \beta_T \int_0^\infty \frac{\ln(1 + e^{\beta_T(\mu - E)})}{1 + e^\theta(E)} dE \quad (39)
\]

- When \(\theta(E)\) approximated by linearized Gamow factor, then \(\theta(E) \to \theta_{ns}(E)\) and \(N(n, s) = N[\theta_{ns}(E)]\).

\[
dN(\theta) = N[\theta(E)]\text{-integrand}
\]

\[
E_m = \text{max} \{dN[\theta(E)]\}
\]

- In \(T\) regime, \(E_m \approx \mu + \phi\)
- In \(F\) regime \(E_m \approx \mu\)
- In \(TF\) regime, \(\beta_T = \beta_F(E_m)\)

Iterative approach to find \(E_m\) (iteration labeled by \(j\)) is now possible in steps such that \(E_m \approx E_5\) typically to 4\(^{th}\) decimal place.

- An accurate iterative approach to finding the location of \(E_m\) is available\(^a\), but here, a simple method is demonstrated.
- 3 points define local parabolic fit, max of which estimates \(E_m\). Process can be improved by re-centering fit points and iterating if desired (but once is good).

\(^a\)Jensen, McDonald, Harris, Shiffler, Cahay, JAP126, 245301 (2019)
Maximum \( (E_m) \) Evaluation III

\[ dJ_{ns}(E)/dJ(E_m): T = 800 \text{ K} \]

\[ dJ(E)/dJ(E_m): T = 800 \text{ K} \]

\[ \log(F \text{ [eV/nm]} \]
Maximum (Em) Evaluation IV

- **Graph 1:**
  - Plots of $n$ vs. $F$ (eV/nm) for different temperatures (50 K, 100 K, 300 K).
  - Lines represent $\beta_T/\beta_F(\mu + \phi)$ and $\beta_T/\beta_F(\mu)$.

- **Graph 2:**
  - Plots of $E_m$ (eV) vs. $\log_{10}(F$ (eV/nm))
  - Lines represent $E_m$, $\mu + \phi$, and $\mu$.

- **Graph 3:**
  - Plots of $dJ(E)/dJ(E_m)$ vs. $E_m$ (eV) for different temperatures (600 K, 1000 K, 1500 K).
  - Lines represent $0.37$, $0.77$, $0.96$, $0.120$, $0.149$, and $0.864$. 

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Sections Outline

1. The Canonical Equations
   - Historical Motivation
   - Conventions
   - Basic Current Density

2. Thermal-Field Emission
   - Original GTF
   - Shape Factor
   - Improvements

3. Usage in Simulation
   - Compound Emitter
   - Hot Fibers and Nanowires
   - Rough Surface
Characterizing current as thermal-like (RLD) or field-like (FN) is precarious when microscale features exist on mesoscale emitters

- For simplicity: on axis protrusion (but methodology allows it anywhere on surface)
- Form compound emitter by placing small dipole atop larger. Dipole potential oriented at polar $\theta_o$ wrt $\hat{z}$-axis is given by
  
  $$U_{dip} = \frac{P (z \cos \theta_o + \rho \sin \theta_o - a)}{[(z - a \cos \theta_o)^2 + (\rho - a \sin \theta_o)]^{3/2}}$$

- $P$ is dipole strength. For primary dipole oriented normal to cathode plane, immersed in background field $F_o$ is
  
  $$U(\rho, z) = -F_o z + U_{dip}(\rho, z) \text{ with } \theta_o \to 0: U(a \sin \theta, a \sin \theta) \equiv 0 \to P = F_o a^3$$
  
  and
  
  $$U(\rho, z) = -F_o z \left[ 1 - \left( \frac{a^2}{z^2 + \rho^2} \right)^{3/2} \right]$$
  
  $\equiv -F_o z + U_o(\rho, z)$ (40)

- Compound emitter: smaller protrusion dipole of strength $\delta P = p(r) F_o a^3$ and radius $ra$ at apex of primary $+$ its image

Effective emitter shape shown by white region; red lines are contours of bare single dipole (primary) with same dipole strength.
Compound Emitter II

- Resulting equipotential not contiguous for $U = 0$ surface: made so with inclusion of small background term $U_c$
- $U(\rho, z)$ vanishes at apex $z_{\text{tip}} = (1 + r)a$: sets $p(r)$
  \[ p = \frac{r + (1 + r^2)^{-3/2} - (1 + r)^{-2}}{r^{-2} + 2(4 + r^2)^{-3/2} - (2 + r)^{-2}} \approx \frac{3}{2} r^3 (2 - 3r) \]  
  (41)
- Choice of $U_c$ satisfies $U(ra, a) = 0$:
  \[ \frac{U_c}{F_o a} = 1 - \frac{1}{(1 + r^2)^{3/2}} + \frac{2p}{(4 + r^2)^{3/2}} \approx \frac{3}{4} r^2 (2 + r) \]  
  (42)
- Exact forms for computation; approximate forms suggest scaling\(^a\)

Field Enhancement: Field $F_s$ along (and normal) to surface

\[ F_s(\rho) \equiv \sqrt{F(\rho, z_{s}(\rho))} + F_{z}(\rho, z_{s}(\rho)) \]  
(43)

Let: $x = \rho/a, y = z_{s}(\rho)/a, f_x = F(\rho, z_{s}(\rho))/F_o$ and $f_y = (F_{z}(\rho, z_{s}(\rho)) - F_o) / F_o, \nu = (0, \pm 1)$:

\[ f_x = \frac{3xy_0}{R_0} - 3p \left( \frac{xy_+}{R_+} - \frac{xy_-}{R_-} \right) \]
\[ f_y = -\frac{x^2 - 2y^2}{R_0} + p \left( \frac{x^2 - 2y^2}{R_+} - \frac{x^2 - 2y^2}{R_-} \right) \]  
(44)

Schottky’s conjecture: $\beta_{\text{tip}} = F_s/F_o = \text{primary } \beta \times \text{protrusion } \beta \text{ (or } 3 \times 3 = 9\text{)}$

Analytic (Padé form suggests behavior):

\[ \frac{F_{\text{tip}}}{F_o} = 1 + \frac{2p}{r^3} + \frac{2}{(1 + r)^{3}} - \frac{2p}{(2 + r)^{3}} \approx \frac{9 - 11.56r + 4.273r^2}{1 + 0.3783r - 1.029r^2} \]  
(45)

departure from $\beta_{\text{tip}} = 9$ as $r$ increases faster than anticipated were protrusions to exhibit "self-similarity" (the Point Charge Model)

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The Canonical Equations
Thermal-Field Emission
Usage in Simulation

Compound Emitter
Hot Fibers and Nanowires
Rough Surface

**Total Current I**

$\beta(\rho) = F[\rho, z_s(\rho)]/F_o$ along surface for $r = 0.1$ and $r = 0.25$; gray line is $\beta(\rho)$ associated with a hemisphere.

Schottky’s conjecture: $\beta_{tip} = F_s/F_o = \text{primary } \beta \times \text{protrusion } \beta$ (or $3 \times 3 = 9$): Departure from $\beta_{tip} = 9$ as $r$ increases faster than for “self-similarity” PCM

$$I(V) \equiv \int_{\Omega} J_{GTF}(F_s, T) \sqrt{1 + (\partial_\rho z_s)^2} \ 2\pi \rho \ d\rho$$

(46)
Tip current $I$ is $J$ integrated over surface $\Omega$

$$I(V) \equiv \int_{\Omega} J_{GTF}(F_s, T) \, dA \quad (47)$$

$$dA = \sqrt{1 + (\partial_\rho z_s)^2} \, 2\pi \rho \, d\rho \quad (48)$$

e.g., primary emitter: $z_s(\rho) = \sqrt{a^2 - \rho^2}$ and $\rho = a \sin \theta$, then $1 + (\partial_\rho z_s)^2 = \sec^2 \theta$ and $dA_h/d\rho = 2\pi a \tan \theta$

Discretize:

$$\partial_\rho z_s \approx \frac{z_s(\rho_{j+1}) - z_s(\rho_j)}{\rho_{j+1} - \rho_j}$$

Discrete surface element:

$$\Delta A_j = \sum_{j=1}^{N} 2\pi \rho_j \sqrt{(\rho_j - \rho_{j-1})^2 + (z_j - z_{j-1})^2} \quad (49)$$

where $\rho_0 = 0$ and $z_0 = (1 + r)a$. The $\sqrt{\ldots}$ term is $dl_j$

of ribbon: Average $J$ along ribbon is

$$\langle J_j \rangle \equiv \frac{1}{2} \left[ J[F_s(\rho_j)] + J[F_s(\rho_{j-1})] \right] \quad (50)$$

Numerical $I(V)$ From Emitter: $U(D, 0) = qV_a$

$$I(V_a) = \sum_{j=1}^{N} \langle J_j \rangle \Delta A_j \quad (51)$$

Limiting cases

$$I_t(V_a) = 2\pi a^2 \int_{0}^{1} J_{RLD}(T, 3F_o x) \, dx \quad (52)$$

$$I_f(V_a) = 2\pi (ra)^2 \int_{0}^{1} J_{FN}(\beta_{tip} F_o x) \, dx \quad (53)$$
Characterization I

- Departures of $J(F)$ from $J_t$ and $J_f$ reveal the onset of TF emission\(^a\)
- Departures of $I(V)$ from $I_t$ and $I_f$ reveal how compound emitter transitions from being a thermal emitter at low fields to being a field emitter at high fields
- Intermediate field regime poorly characterized by such a dichotomy: T-dominated processes (on primary) co-exist with F-dominated processes (on protrusion) on same compound structure
- Comparison for $r = (0.1, 0.25) = (\circ, \bullet)$
  - $I_f$ (dark / light) blue for $(\circ, \bullet)$
  - $I_t$ (in red) due to primary
- Factor of $5/4$ associated with base for $I_t$ brings RLD agreement with GTF at low $F$.

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Fowler-Nordheim using Spindt $v(y), t(y)$ (sp):

$$I(F_o) = A_o J_{sp}(F_o)$$

(54)

- **fit** to $\ln(I/F_o^2) = Y - S/F_o$
- infer $\beta, A_o$

$$\beta = \frac{4V_o \sqrt{2m\Phi^3}}{3\hbar S}$$

$$A_o = \frac{e^Y}{A_{sp}\beta^2}$$

- Numerical: $r = 0.10$, $\beta = 7.68, 8411 < A(F_o) < 9762$
  YS: $\beta = 9.21$ and $A_o = 907$ nm$^2$

- Numerical: $r = 0.25$, $\beta = 6.19, 59383 < A(F_o) < 50455$
  YS: $\beta = 8.13$ and $A_o = 1912$ nm$^2$

- $\beta$ over by $20\% - 30\%$, $A_o$ under by $10\% - 30\%$

\(^a\) all units in (eV, nm, fs, q)
Richardson Eq.

\[ I_{RLD}(F_o) = A_c J_{RLD}(T) \]  \hspace{1cm} (55)

Calibrate:

\[ \beta_{eff} = \frac{(k_B T)^2}{4Q} \left[ \frac{\ln(I_a/I_b)}{\sqrt{F_a} - \sqrt{F_b}} \right]^2 \]

- fit\(^a\) to \( \ln(I/T^2) = Y - S/T \)

\[ \Phi_{eff} = k_B S + \sqrt{4\beta_{eff} F_o Q} \]

\[ A_{eff} = \frac{e^Y}{A_{RLD}} \]

- Numerical: \( \beta = 3, \Phi = 2.6 \text{ eV}, A/\pi a^2 = 1 + r^2 + (9/4) \)

- YS (0.10):
  \( \beta = 2.12, \Phi = 2.26, A/\pi a^2 = 0.443 \)

- YS (0.25):
  \( \beta = 2.38, \Phi = 2.27, A/\pi a^2 = 0.425 \)

\(^a\) all units in (eV, nm, fs, q)
Carbon fiber field emission sources for HPM and pulsed power applications: inherent resistivity of fiber causes substantial $T$-variation along length, and TF emission at apex, analogous to large $T$ increases in CNT’s.

Typical fiber: length of 1.5 mm; diameter of 35 $\mu$m. After operation, apex deformed, side damaged due to thermal processes.

Simple heating model: analytic $0^{th}$ order approximation of $T$-variation variation; can be corrected by simple numerical procedure.

Model implemented into MICHELLE beam optics code: self-consistent prediction of $J$ and $T$ along a fiber.

Temperature along rod of length $L$, radius $r$ and attached to base at temperature $T_o$:

$$\frac{d}{dx} \left( \kappa \frac{dT}{dx} \right) - \frac{2\sigma_B}{r} \left( T^4 - T_o^4 \right) = -\frac{I_o^2}{\pi^2 r^4 \sigma(T)} \tag{56}$$

$\kappa(T)$ thermal conductivity, $\sigma_B T^4$ radiative heat loss

$\sigma(T)$ electrical conductivity, $I_o$ current passing from apex

**Electrical conductivity in Drude model for a relaxation time $\tau(T)$ (dominated by $e - p$)**

$$\sigma(T) = \frac{q^2}{m} \rho \tau(T); \quad \tau_{ep}(T) = \frac{\hbar}{2\pi \lambda \kappa_B T} \tag{57}$$

Bohm-Staver: $\lambda = 1/2$, but treated as adjustable

**$\kappa(T)$ related to $\sigma(T)$ by $\kappa = L_o T \sigma(T)$**

$$\kappa(T) = \frac{\pi^2}{3} \left( \frac{k_B}{q} \right)^2 T; \quad \sigma(T) = \frac{\pi \hbar k_B \rho}{6 \lambda m} \tag{58}$$

Form of $\tau_{ep}(T)$ makes $\sigma(T)$ constant

As a result $\partial_x (\kappa \partial_x T) \rightarrow \kappa \partial_x^2 T$ in Eq. (56).
The Canonical Equations
Thermal-Field Emission
Usage in Simulation

Thermal-Field Emission from Carbon Fibers III

Introduce constants

\[ \omega_o = \frac{2 \sqrt{3} \lambda I_o L m}{\pi qhr^2 \rho}; \quad \gamma \equiv \left[ \frac{4 \pi \lambda mL^2 (k_B T_o)^3}{5h^4 c^2 r^2 \rho} \right]^{1/2} \] (59)

Dimensionless coordinates:
- \( L \) is length of fiber, \( T_o \) is temperature of base
- \( x = sL; \quad T = (t + 1)T_o \)

Eq. (56) becomes

\[ \frac{d^2}{ds^2} t = \frac{\gamma^2}{4} \left[ (1 + t)^4 - 1 \right] - \omega_o^2 (1 + t) \] (60)

Separate out red “Harmonic Oscillator” part:

\[ t(s) \equiv f(s) + \gamma^2 \Delta(s): \]

\[ \frac{d^2}{ds^2} f = - \omega_o^2 (1 + f) \rightarrow \]

\[ f(s) = \frac{\cos[\omega_o (1 - s)]}{\cos \omega_o} - 1 \] (61)

Singularity: \( I_{\text{max}} \) when \( \omega_o = \pi/2 \), or

\[ I_{\text{max}} = \frac{\sqrt{3} \pi qr^2}{2k_B L} \kappa = \frac{\sqrt{3} \pi^2 \hbar q \rho r^2}{12 \lambda mL} \] (62)
Thermal-field emission from carbon fiber modeled in MICHELLE PIC code. Macro-particle green trajectories contain over an order of magnitude more charge than red trajectories. Majority of the current from fillet (field enhancement largest). Rapid rise in $T$ results in significant TF and thermal emission.

3 mm parallel plate gap, radius of 15 mm. Fiber is 35 $\mu$m in diameter and 1.5 mm tall. Fillet on emitter cap is a 2 $\mu$m radius. Base at 300 K. Electrons emitted from red-colored region. DOI: 10.1063/5.0044800

$T$ [K] vs. distance along fiber for AK-Gap voltage shown. Lower curves are stable solutions; upper curve enters runaway (does not converge at an intermediate state on way to failure).
Mean Transverse Energy from Roughness I

- $F = q|\mathcal{E}|$ is a force: Total force $F^2 = F_x^2 + F_y^2 + F_z^2$ is product of $\beta(x, y, z_s)$ with the background $F_o$ such that $|\vec{F}| = \beta F_o$.
- $\beta(x, y, z_s)$ for surface used to find $|\vec{F}| = \beta F_o$ used in $J_{GTFP}(F, T)$
- ($\beta > 1$) near apexes of protrusions, ($\beta < 1$) occurs in valleys between protrusions
Ballistic Impulse Approximation replaces the launch $v_o$ from surface with varying field with an impulse-modified $v$ in a constant (ballistic) field. Impulse factor $\sigma$ alters $\vec{v}_o$ (⊥ to surface):

$$\sigma(x, y) \rightarrow 1 + \frac{1}{2} C \kappa \beta(x, y)$$  \hspace{1cm} (63)

Analytic approx $(v_x, v_y, v_z)$ (in units of $|\vec{v}_o|$) at $z = z_a$ is then specified by the sequence:

- Find surface $z_s(x, y)$ using the PCM
- Evaluate $\beta(x, y)$ at $z_s$
- Specify $\sigma$ via Eq. (63)
- Evaluate $\vec{v}(t)$ over surface
- Specify the time $t_a$, defined by $z(t_a) = z_a \sim 2z_c$
- Weight $\vec{v}(t_a)$ by emission area element and construct $v$-distributions for MTE evaluation

If successful, simplifies PIC Emitter model\(^a\)

\(^a\)K.L. Jensen, M. McDonald, et al., JAP 125, 234303 (2019).
Mean Transverse Energy from Roughness III

Particle trajectories simulated in MICHELLE
1 in 20 particles are shown.

1 Periodic boundary
2 Cathode with rough surface
3 Vacuum space and trajectories
4 Vacuum space outline (cut at y = 0). (bottom) Cathode surface showing grid.

Agreement with BIA demonstrated
Mean Transverse Energy from Roughness IV

Top-down view of cathode (red ◦), showing 256 emitted $e^-$ trajectories over one cycle. $\mathbf{B}$-field normal to surface. Origin position same in each quadrant. Label is ratio of max orbital radius with radius of cathode.

Beam Radius encloses all blue circles

Launch sites at edge of emitter: cold trajectories are normal to surface ($v_\perp = 0$). hot trajectories have $v_\perp$ MB-distributed

Observe larger beam diameter for hot launch conditions

$^a$Figures courtesy of J. Petillo, Leidos
Radius $R(z)$ of beam governed by Beam Envelope equation

$$\frac{d^2}{dz^2} R + \left( \frac{qB}{2\beta\gamma mc} \right)^2 R - \frac{2I_a}{(\gamma\beta)^3 I_o} \frac{1}{R} - \frac{\varepsilon^2}{R^3} = 0$$  \hspace{1cm} (64)

- Magnetic term $B = \text{mag field}$; Space Charge term $I_a = \text{beam current}$
- $I_o = qmc^2/\alpha\hbar = 17.045 \text{ kA}$ is Alfven-Lawson current
- Relativistic: $\beta = v/c$; $\gamma = 1/\sqrt{1 - \beta^2} \rightarrow \gamma mc^2 = mc^2 + V_b$ (acceleration potential)
- $f = \text{frequency}, 2\pi fnR/c = \text{zero of Bessel function, } n = \text{mode}$

Brillouin (or “smooth”) flow: sum of $B$, SC, and $\varepsilon$ terms vanish:

$$\frac{I_a}{\pi R^2} = \frac{I_o}{8\pi} \left( \frac{2V_b}{mc^2} \right)^{1/2} \left( \frac{qB}{mc} \right)^2 \left\{ 1 - \frac{8mV_b\varepsilon^2}{q^2B^2R^4} \right\}$$  \hspace{1cm} (65)

$$J_{beam}(\varepsilon) = J_{beam}(0) \{ 1 - \delta(\varepsilon) \}$$  \hspace{1cm} (66)

- Technology constrains $|B|$; $V_b$ depends on platform; Higher $f \leftrightarrow$ lower $R$
- Emittance affects ability to bunch and focus beam prior to power extraction
CONCLUDING REMARKS

**Canonical Equations and General Thermal Field Emission**

- Designed for rapid and repeated evaluation of emission from mesoscale structures with microscale protrusions
- ...to meet the needs of interpreting characterization measurements, prediction of performance from compound emitters, and providing models that can profitably be used in the simulation of electron beams
- Lorentzian correction accounts for departures of \( \theta(E) \) from its linear approximation

**Impact On Usage and Characterization**

- Compound emitters include geometry and roughness is a combination of point dipoles describing the primary base and the apex protrusion and is analytic
- Rapid analytical models of TF current density and temperature variation needed for millions of elements on typical surface
- GTF contrasted to RLD and FN: Impact on \( I(V, T) \) undermines characterization - should include TF emission, local \( F \) and \( T \) variation, physical geometry
- Consequences on characterization of emitters, simulation in devices using density modulated beams, and particle-in-cell (PIC) codes are expected